# Set Theory Revisited

# Outline for Today

- ***Proofs on Sets***
  - Making our intuitions rigorous.
- ***Formal Set Definitions***
  - What do our terms mean?
- ***Appendices: Examples***
  - Sample proofs to help you get the hang of the ideas here.

# Recap from Last Time

| | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|
| $\forall x.\ A$ | Initially, ***do nothing***. Once you find a $z$ through other means, you can state it has property $A$. | Have the reader pick an arbitrary $x$. We then prove $A$ is true for that choice of $x$. |
| $\exists x.\ A$ | Introduce a variable $x$ into your proof that has property $A$. | Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$. |
| $A \to B$ | Initially, ***do nothing***. Once you know $A$ is true, you can conclude $B$ is also true. | Assume $A$ is true, then prove $B$ is true. |
| $A \land B$ | Assume $A$. Also assume $B$. | Prove $A$. Also prove $B$. |
| $A \lor B$ | Consider two cases.<br>Case 1: $A$ is true.<br>Case 2: $B$ is true. | Either prove $\neg A \to B$ or prove $\neg B \to A$.<br>*(Why does this work?)* |
| $A \leftrightarrow B$ | Assume $A \to B$ and $B \to A$. | Prove $A \to B$ and $B \to A$. |
| $\neg A$ | Simplify the negation, then consult this table on the result. | Simplify the negation, then consult this table on the result. |

# New Stuff!

# Proving Results from Set Theory

***Claim:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that $A$ is contained in $B$. Since $B \in C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \in C$, as required. ■

***Claim:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

Which (if any) of these claims are true?
Answer at

***https://cs103.stanford.edu/pollev***

***Claim:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that $A$ is contained in $B$. Since $B \in C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \in C$, as required. ■

This claim is not true! For example…

$$\emptyset \in \{\emptyset\}$$
$$\{\emptyset\} \in \{\{\emptyset\}\}$$
$$\emptyset \notin \{\{\emptyset\}\}$$

**Lie:** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

**Bad Proof:** Assume $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that $A$ is contained in $B$. Since $B \in C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \in C$, as required. ■

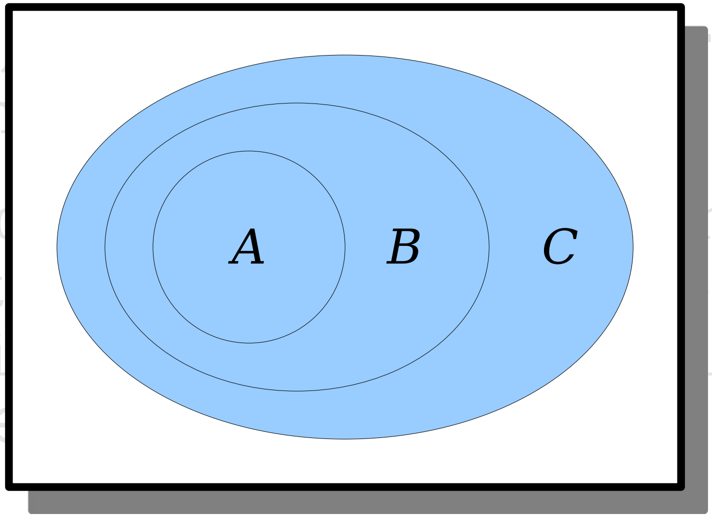
This claim is not true! For example…

$$\emptyset \in \{\emptyset\}$$
$$\{\emptyset\} \in \{\{\emptyset\}\}$$
$$\emptyset \notin \{\{\emptyset\}\}$$

***Lie:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Bad Proof:*** Assume $A$, … $A \in B$ and $B \in C$.
We need to show th…

Since $A \in B$, we kn… $B$. Since $B \in C$, we know that $B$ is cont… because $A$ is contained in $B$ and $B$ is contai… $A$ is contained is $C$.
This means that $A$ …




***Claim:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

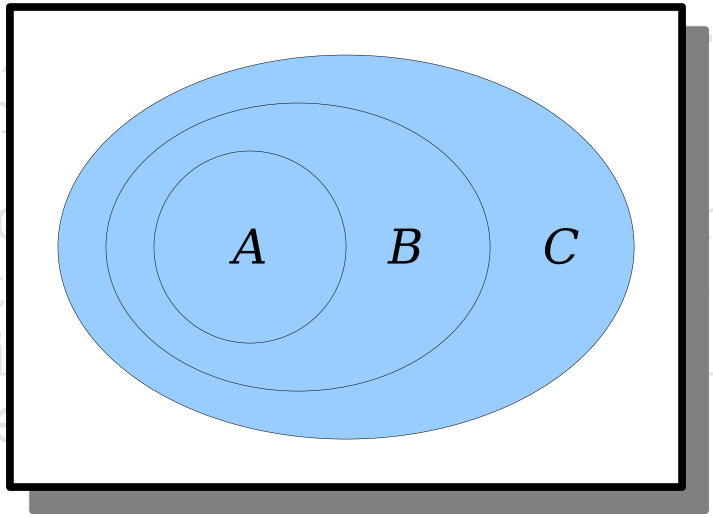
***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$.
We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Lie:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Bad Proof:*** Assume $A$, … e $A \in B$ and $B \in C$.
We need to show th…

Since $A \in B$, we kn… n $B$. Since $B \in C$, we
know that $B$ is cont… ecause $A$ is contained.
in $B$ and $B$ is contai… $A$ is contained is $C$.
This means that $A \in$ …




***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$.
We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Lie:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$.
We need to show that $A \in C$.

Since $A \in B$, we know that $A$ is contained in $B$. Since $B \in C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \in C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof (?):*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$.
We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Lie:*** If $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$, then $A \in C$.

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \in B$ and $B \in C$. We need to show that $A \in C$.

Since $A \in B$, we know that $A$ is contained in $B$. Since $B \in C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \in C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

This can't be a good proof; the same basic argument proves a false claim!

***Claim:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

***Proof (?):*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Claim:*** Let $A$, $B$, and $C$ be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

***Proof (?):*** Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of $A$ are in $B$ and there are other elements of $B$. Since $A \subsetneq C$, all elements of $A$ are also in $C$ and there are other elements of $C$. Therefore, all elements of $A$ are in both $B$ and $C$, and there are some other elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■

*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

Which (if any) of these claims are true?
Answer at

***https://cs103.stanford.edu/pollev***

**Claim:** Let $A$, … e
have $A \subsetneq B$ …

**Proof (?):** Ass… $B \cap C$.

Since $A \subsetneq B$ …
elements o… are in
both $B$ and …

This claim is not true! For example…

$$\{137\} \subsetneq \{137, 103\}$$
$$\{137\} \subsetneq \{137, 107\}$$
$$\text{But } \{137\} = \{137, 103\} \cap \{137, 107\}$$

**Claim:** Let $A$, $B$, and $C$ be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

**Proof (?):** Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of $A$ are in $B$ and there are other elements of $B$. Since $A \subsetneq C$, all elements of $A$ are also in $C$ and there are other elements of $C$. Therefore, all elements of $A$ are in both $B$ and $C$, and there are some other elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■

*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

This claim is not true! For example…

$$\{137\} \subsetneq \{137, 103\}$$
$$\{137\} \subsetneq \{137, 107\}$$
$$\text{But } \{137\} = \{137, 103\} \cap \{137, 107\}$$

***Lie:*** Let $A$, $B$, and $C$ be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

***Bad Proof:*** Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of $A$ are in $B$ and there are other elements of $B$. Since $A \subsetneq C$, all elements of $A$ are also in $C$ and there are other elements of $C$. Therefore, all elements of $A$ are in both $B$ and $C$, and there are some other elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■

*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

***Claim:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

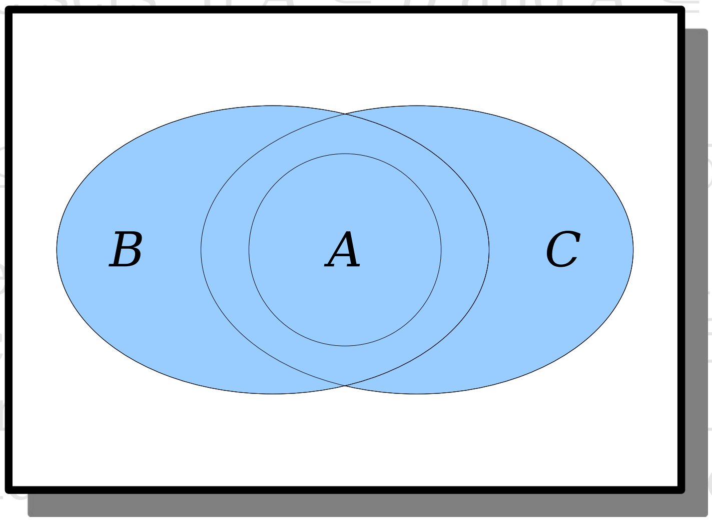
***Proof (?):*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Lie:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subsetneq B \cap C$.

***Bad Proof:*** Assume $A$ … d to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all ele … there are other
elements of $B$. Sinc … f $A$ are also in $C$ and
there are other ele … ll elements of $A$ are
in both $B$ and $C$, an … elements in $B$ and $C$.
Therefore, we see that $A \subsetneq B \cap C$. ■




*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

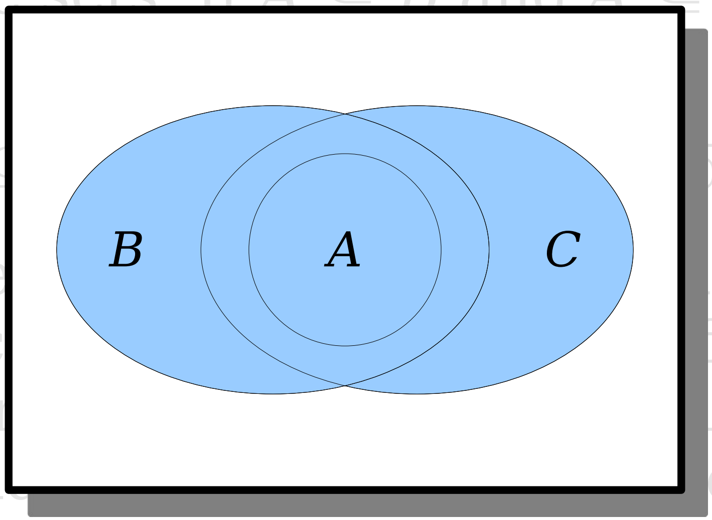
***Proof (?):*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Lie:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subsetneq B \cap C$.

***Bad Proof:*** Assume $A$ … d to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all ele… there are other elements of $B$. Sinc… f $A$ are also in $C$ and there are other ele… ll elements of $A$ are in both $B$ and $C$, an… elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■




*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

***Proof (?):*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Lie:*** Let $A$, $B$, and $C$ be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

***Bad Proof:*** Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of $A$ are in $B$ and there are other elements of $B$. Since $A \subsetneq C$, all elements of $A$ are also in $C$ and there are other elements of $C$. Therefore, all elements of $A$ are in both $B$ and $C$, and there are some other elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■

*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then we have $A \subseteq B \cap C$.

***Bad Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Lie:*** Let $A$, $B$, and $C$ be sets. If $A \subsetneq B$ and $A \subsetneq C$, then we have $A \subsetneq B \cap C$.

***Bad Proof:*** Assume $A \subsetneq B$ and $A \subsetneq C$. We need to show $A \subsetneq B \cap C$.

Since $A \subsetneq B$, all elements of $A$ are in $B$ and there are other elements of $B$. Since $A \subsetneq C$, all elements of $A$ are also in $C$ and there are other elements of $C$. Therefore, all elements of $A$ are in both $B$ and $C$, and there are some other elements in $B$ and $C$. Therefore, we see that $A \subsetneq B \cap C$. ■

*(Reminder: $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.)*

# What Went Wrong?

- The style of arguments you've just seen are ***not*** how to prove results on sets.
- As you've seen:
  - The reliance on high-level terms like "contained" is not mathematically precise.
  - A discussion of "all elements" of a set is not how to reason about collections of objects.
- ***Question:*** How do we write rigorous proofs about sets?

# The Importance of Definitions

- As you've seen this week, formal definitions underpin mathematical proofs.
- The major issue from the previous proofs is that we haven't defined what our terms mean.
  - How do we define what $A \in B$ means?
  - How do we define what $A \subseteq B$ means?
  - How do we define what $A \cap B$ means?
- Think back to our proof triangle: we currently have intuitions for these concepts, but not formal definitions.

| | Is defined as... | If you ***assume*** this is true... | To ***prove*** that this is true... |
|---|---|---|---|
| $S \subseteq T$ | | | |
| $S = T$ | | | |
| $x \in S \cap T$ | | | |
| $x \in S \cup T$ | | | |
| $X \in \wp(S)$ | | | |
| $x \in \{ y \mid P(y) \}$ | | | |

# Proofs on Subsets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

# Defining Subsets

- Formally speaking, if $S$ and $T$ are sets, we say that $S \subseteq T$ when the following holds:

$$\forall x \in S.\ x \in T$$

- Now, suppose you're working with a proof where you encounter $S \subseteq T$. Think back to the proof table.
  - To ***assume*** that $S \subseteq T$, what should you do?
  - To ***prove*** that $S \subseteq T$, what should you do?

Answer at

***https://cs103.stanford.edu/pollev***

| | Is defined as... | If you ***assume*** this is true... | To ***prove*** that this is true... |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |
| $S = T$ | | | |
| $x \in S \cap T$ | | | |
| $x \in S \cup T$ | | | |
| $X \in \wp(S)$ | | | |
| $x \in \{\ y \mid P(y)\ \}$ | | | |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:***

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$.

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$.

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$.
We need to prove that $A \subseteq C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
| --- | --- | --- | --- |
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required.

# A Correct Proof on Sets

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

---

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

---

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

---

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

***Theorem:*** If $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

***Proof:*** Let $A$, $B$, and $C$ be sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, pick some $x \in A$; we need to show that $x \in C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, we see that $x \in C$, as required. ■

---

***Bad Proof:*** Assume $A$, $B$, and $C$ are sets where $A \subseteq B$ and $B \subseteq C$. We need to show that $A \subseteq C$.

Since $A \subseteq B$, we know that $A$ is contained in $B$. Since $B \subseteq C$, we know that $B$ is contained in $C$. Therefore, because $A$ is contained in $B$ and $B$ is contained in $C$, we know that $A$ is contained is $C$. This means that $A \subseteq C$, as required. ■

# Unions and Intersections

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

# Unions and Intersections

- The statement $x \in S \cap T$ is defined as

$$\boldsymbol{x \in S \quad \wedge \quad x \in T.}$$

- The statement $x \in S \cup T$ is defined as

$$\boldsymbol{x \in S \quad \vee \quad x \in T.}$$

- These are operational definitions: they show how unions and intersections interact with the ∈ relation rather than saying what the union or intersection of two sets "are."

| | Is defined as... | If you ***assume*** this is true... | To ***prove*** that this is true... |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S. \ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |
| $S = T$ | | | |
| $x \in S \cap T$ | $x \in S \ \land \ x \in T$ | Assume $x \in S$. Then assume $x \in T$. | Prove $x \in S$. Also prove $x \in T$. |
| $x \in S \cup T$ | $x \in S \ \lor \ x \in T$ | Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$. | Either prove $x \in S$ or prove $x \in T$. |
| $X \in \wp(S)$ | | | |
| $x \in \{ y \mid P(y) \}$ | | | |

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:***

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $x \in S \cap T$ | $x \in S \land x \in T$ | Assume $x \in S$. Then assume $x \in T$. | Prove $x \in S$. Also prove $x \in T$. |

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required.

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required. ■

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required. ■

---

***Bad Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

***Theorem:*** Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

***Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to prove that $A \subseteq B \cap C$. So pick some $x \in A$; we need to show that $x \in B \cap C$.

Since $x \in A$ and $A \subseteq B$, we know $x \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Therefore, we see that $x \in B$ and $x \in C$, so $x \in B \cap C$, as required. ■

---

***Bad Proof:*** Assume $A \subseteq B$ and $A \subseteq C$. We need to show $A \subseteq B \cap C$.

Since $A \subseteq B$, all elements of $A$ are in $B$. Since $A \subseteq C$, all elements of $A$ are also in $C$. Therefore, all elements of $A$ are in both $B$ and $C$. Therefore, we see that $A \subseteq B \cap C$. ■

# Set Equality

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

| | Is defined as... | If you ***assume*** this is true... | To ***prove*** that this is true... |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |
| $S = T$ | $S \subseteq T \land T \subseteq S$ | Assume $S \subseteq T$ and $T \subseteq S$. | Prove $S \subseteq T$. Also prove $T \subseteq S$. |
| $x \in S \cap T$ | $x \in S \land x \in T$ | Assume $x \in S$. Then assume $x \in T$. | Prove $x \in S$. Also prove $x \in T$. |
| $x \in S \cup T$ | $x \in S \lor x \in T$ | Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$. | Either prove $x \in S$ or prove $x \in T$. |
| $X \in \wp(S)$ | | | |
| $x \in \{ y \mid P(y) \}$ | | | |

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** See appendix!

# Time-Out for Announcements!

# Problem Set Three

- Problem Set Two was due today at 1:00PM.
  - Need more time? You can use a late day to extend the deadline to Saturday at 1:00PM.
- Problem Set Three goes out today and is due next Friday at 1:00PM.
  - ***Start early!*** This gives you more time to make slow and steady progress.
  - Use the techniques from this week: the assume/prove table, the two-column organizer, etc.

Did you know that 20% of TAs
are lonely in their office hours?

Stop by Sunday or Monday OH! Get your questions answered, meet cool people, have a good time.

Back to CS103!

# Set-Builder Notation

# Set-Builder Notation

- Let $S$ be the set defined here:

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \geq 137\ \}$$

- Now imagine you have some quantity $x$. Based on this…
  - … if you ***assume*** that $x \in S$, what does that tell you about $x$?
  - … if you ***prove*** that $x \in S$, what do you need to prove?

Answer at

***https://cs103.stanford.edu/pollev***

# Set-Builder Notation

- Like unions and intersections, we have an operational definition for set-builder notation. It's the following:

**Let $S = \{ y \mid P(y) \}$.**
**Then $x \in S$ when $P(x)$ is true.**

- So, for example:
  - $x \in \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$ means $x \in \mathbb{N}$ and $x$ is even.
  - $x \in \{ n \mid \exists k \in \mathbb{N}.\ n = 2k + 1\}$ means that there is a $k \in \mathbb{N}$ where $x = 2k + 1$. (Equivalently, $x$ is an odd natural number)
- ***Key Point:*** The placeholder variable disappears in all these examples. After all, *it's just a placeholder.*

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |
| $S = T$ | $S \subseteq T \land T \subseteq S$ | Assume $S \subseteq T$ and $T \subseteq S$. | Prove $S \subseteq T$. Also prove $T \subseteq S$. |
| $x \in S \cap T$ | $x \in S \land x \in T$ | Assume $x \in S$. Then assume $x \in T$. | Prove $x \in S$. Also prove $x \in T$. |
| $x \in S \cup T$ | $x \in S \lor x \in T$ | Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$. | Either prove $x \in S$ or prove $x \in T$. |
| $X \in \wp(S)$ | | | |
| $x \in \{ y \mid P(y) \}$ | $P(x)$ | Assume $P(x)$. | Prove $P(x)$. |

# Proofs on Set-Builder Notation

# Some Useful Notation

- If $n$ is a natural number, we define the set **$[n]$** as follows:

$$[n] = \{ k \mid k \in \mathbb{N} \ \land \ k < n \}$$

- So, for example:
  - $[3] = \{0, 1, 2\}$
  - $[0] = \emptyset$
  - $[5] = \{0, 1, 2, 3, 4\}$

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

**Theorem:** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

| *What We're Assuming* | *What We Need to Prove* |
|---|---|
| $m \in \mathbb{N}$<br>$n \in \mathbb{N}$<br>$m < n$<br>$[z] = \{ k \mid k \in \mathbb{N} \wedge k < z \}$ | $[m] \subseteq [n]$ |

**Theorem:** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

| *What We're Assuming* | *What We Need to Prove* |
| --- | --- |
| $m \in \mathbb{N}$ | ~~$[m] \subseteq [n]$~~ |
| $n \in \mathbb{N}$ | $\forall x \in [m].\ x \in [n]$ |
| $m < n$ | |
| $[z] = \{\ k \mid k \in \mathbb{N} \land k < z\ \}$ | |

**Theorem:** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

| *What We're Assuming* | *What We Need to Prove* |
|---|---|
| $m \in \mathbb{N}$ | ~~$[m] \subseteq [n]$~~ |
| $n \in \mathbb{N}$ | ~~$\forall x \in [m].$~~ $x \in [n]$ |
| $m < n$ | |
| $[z] = \{ k \mid k \in \mathbb{N} \land k < z \}$ | |
| $x \in [m]$ | |

**Theorem:** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

| What We're Assuming | What We Need to Prove |
|---|---|
| $m \in \mathbb{N}$ | ~~$[m] \subseteq [n]$~~ |
| $n \in \mathbb{N}$ | ~~$\forall x \in [m].$~~ $x \in [n]$ |
| $m < n$ | |
| $[z] = \{ k \mid k \in \mathbb{N} \land k < z \}$ | |
| | |
| $x \in [m]$ | |
| $x \in \mathbb{N}$ | |
| $x < m$ | |

**Theorem:** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$

| *What We're Assuming* | *What We Need to Prove* |
|---|---|
| $m \in \mathbb{N}$ | ~~$[m] \subseteq [n]$~~ |
| $n \in \mathbb{N}$ | ~~$\forall x \in [m].\ x \in [n]$~~ |
| $m < n$ | $x \in \mathbb{N}$ |
| $[z] = \{ k \mid k \in \mathbb{N} \land k < z \}$ | $x < n$ |
| | |
| $x \in [m]$ | |
| $x \in \mathbb{N}$ | |
| $x < m$ | |

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:***

***Theorem:*** If $m$, $n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$. Then, because $x < m$ and $m < n$, we know that $x < n$.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$. Then, because $x < m$ and $m < n$, we know that $x < n$. Collectively this means that $x \in \mathbb{N}$ and $x < n$, so $x \in [n]$, as required.

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$. Then, because $x < m$ and $m < n$, we know that $x < n$. Collectively this means that $x \in \mathbb{N}$ and $x < n$, so $x \in [n]$, as required. ■

***Theorem:*** If $m, n \in \mathbb{N}$ and $m < n$, then $[m] \subseteq [n]$.

***Proof:*** Assume $m$ and $n$ are natural numbers where $m < n$. We need to show that $[m] \subseteq [n]$. To do so, pick some $x \in [m]$. We'll prove that $x \in [n]$.

Since $x \in [m]$, we know that $x \in \mathbb{N}$ and $x < m$. Then, because $x < m$ and $m < n$, we know that $x < n$. Collectively this means that $x \in \mathbb{N}$ and $x < n$, so $x \in [n]$, as required. ■

Notice that *there is no set-builder notation in this proof*. We were able to avoid it by using the rules for what $x \in \{ y \mid P(y) \}$ say to do.

| | Is defined as… | If you ***assume*** this is true… | To ***prove*** that this is true… |
|---|---|---|---|
| $S \subseteq T$ | $\forall x \in S.\ x \in T$ | Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$. | Ask the reader to pick an $x \in S$. Then prove $x \in T$ |
| $S = T$ | $S \subseteq T \land T \subseteq S$ | Assume $S \subseteq T$ and $T \subseteq S$. | Prove $S \subseteq T$. Also prove $T \subseteq S$. |
| $x \in S \cap T$ | $x \in S \land x \in T$ | Assume $x \in S$. Then assume $x \in T$. | Prove $x \in S$. Also prove $x \in T$. |
| $x \in S \cup T$ | $x \in S \lor x \in T$ | Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$. | Either prove $x \in S$ or prove $x \in T$. |
| $X \in \wp(S)$ | $X \subseteq S$. | Assume $X \subseteq S$. | Prove $X \subseteq S$. |
| $x \in \{\ y \mid P(y)\ \}$ | $P(x)$ | Assume $P(x)$. | Prove $P(x)$. |

# Your Action Items

- ***Read "Guide to Proofs on Discrete Structures."***
  - There's additional guidance and practice on the assume/prove dichotomy and how it manifests in problem-solving.
- ***Read "Discrete Structures Proofwriting Checklist."***
  - Keep the items here in mind when writing proofs. We'll use this when grading your problem set.
- ***Read "Guide to Proofs on Sets."***
  - There's some good worked examples in there to supplement today's lecture, several of which will be relevant for the problem set.
- ***Start Problem Set 3.***
  - Start early and make slow and steady progress.

# Next Time

- ***Graph Theory***
  - A ubiquitous, powerful abstraction with applications throughout computer science.
- ***Vertex Covers***
  - Making sure tourists don't get lost.
- ***Independent Sets***
  - Helping the recovery of the California Condor.

# ***Appendix:*** More Sample Set Proofs

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

| *What I'm Assuming* | *What I Need to Show* |
| --- | --- |
| $A \subseteq C$ | $A \cup B \subseteq C \cup D$ |
| $\forall z \in A.\ z \in C$ | |
| $B \subseteq D$ | |
| $\forall z \in B.\ z \in D$ | |

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

| *What I'm Assuming* | *What I Need to Show* |
|---|---|
| $A \subseteq C$ | ~~$A \cup B \subseteq C \cup D$~~ |
| $\forall z \in A.\ z \in C$ | $\forall x \in A \cup B.\ x \in C \cup D$ |
| $B \subseteq D$ | |
| $\forall z \in B.\ z \in D$ | |

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

| *What I'm Assuming* | *What I Need to Show* |
|---|---|
| $A \subseteq C$ | ~~$A \cup B \subseteq C \cup D$~~ |
| $\forall z \in A.\ z \in C$ | ~~$\forall x \in A \cup B.$~~ $x \in C \cup D$ |
| $B \subseteq D$ | |
| $\forall z \in B.\ z \in D$ | |
| $x \in A \cup B$ | |

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

| *What I'm Assuming* | *What I Need to Show* |
| --- | --- |
| $A \subseteq C$ | ~~$A \cup B \subseteq C \cup D$~~ |
| $\forall z \in A.\ z \in C$ | ~~$\forall x \in A \cup B.\ x \in C \cup D$~~ |
| $B \subseteq D$ | $x \in C$ or $x \in D$ |
| $\forall z \in B.\ z \in D$ | |
| $x \in A \cup B$ | |

**Theorem:** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

| *What I'm Assuming* | *What I Need to Show* |
| --- | --- |
| $A \subseteq C$ | ~~$A \cup B \subseteq C \cup D$~~ |
| $\forall z \in A.\ z \in C$ | ~~$\forall x \in A \cup B.\ x \in C \cup D$~~ |
| $B \subseteq D$ | $x \in C$ or $x \in D$ |
| $\forall z \in B.\ z \in D$ | |
| $x \in A \cup B$ | |
| *Case 1:* $x \in A$ | |
| *Case 2:* $x \in B$ | |

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:***

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

*Case 1:* $x \in A$.

*Case 2:* $x \in B$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

*Case 1:* $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

*Case 2:* $x \in B$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

*Case 1:* $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

*Case 2:* $x \in B$. Then because $B \subseteq D$ and $x \in B$ we have $x \in D$, so $x \in C \cup D$.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

*Case 1:* $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

*Case 2:* $x \in B$. Then because $B \subseteq D$ and $x \in B$ we have $x \in D$, so $x \in C \cup D$.

In either case, we see that $x \in C \cup D$, as required.

***Theorem:*** Let $A$, $B$, $C$, and $D$ be sets where $A \subseteq C$ and $B \subseteq D$. Then $A \cup B \subseteq C \cup D$.

***Proof:*** Pick some $x \in A \cup B$; we need to show that $x \in C \cup D$.

Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider each case separately.

*Case 1:* $x \in A$. Since $A \subseteq C$ and $x \in A$, we see that $x \in C$, and therefore that $x \in C \cup D$.

*Case 2:* $x \in B$. Then because $B \subseteq D$ and $x \in B$ we have $x \in D$, so $x \in C \cup D$.

In either case, we see that $x \in C \cup D$, as required. ■

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:***

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$.

Case 2: $x \in B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

(⇒) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$.

(⇐) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

(⇒) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$.

(⇐) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$. So pick an $x \in A$; we need to show that $x \in B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$. So pick an $x \in A$; we need to show that $x \in B$.

Since $x \in A$, we know that $x \in A \cup B$.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

($\Rightarrow$) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

($\Leftarrow$) Assume $A \cup B = B$. We need to show that $A \subseteq B$. So pick an $x \in A$; we need to show that $x \in B$.

Since $x \in A$, we know that $x \in A \cup B$. And since $x \in A \cup B$ and $A \cup B = B$, we see that $x \in B$, as required.

***Theorem:*** Let $A$ and $B$ be sets. Then $A \subseteq B$ if and only if $A \cup B = B$.

***Proof:*** We will prove each direction of implication.

(⇒) Assume $A \subseteq B$. We need to show that $A \cup B = B$. To do so, we need to show that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$.

First, we'll show $A \cup B \subseteq B$. Pick an $x \in A \cup B$. We need to show that $x \in B$. Since $x \in A \cup B$, we consider two cases:

Case 1: $x \in A$. Then since $x \in A$ and $A \subseteq B$, we have $x \in B$.

Case 2: $x \in B$. Then by assumption $x \in B$.

Either way, we have $x \in B$, which is what we needed to show.

Next, we'll prove $B \subseteq A \cup B$. Pick some $x \in B$. Since $x \in B$, we know that $x \in A \cup B$, as required.

(⇐) Assume $A \cup B = B$. We need to show that $A \subseteq B$. So pick an $x \in A$; we need to show that $x \in B$.

Since $x \in A$, we know that $x \in A \cup B$. And since $x \in A \cup B$ and $A \cup B = B$, we see that $x \in B$, as required. ■

***Theorem:*** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

***Theorem:*** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

| *What I'm Assuming* | *What I Need to Show* |
| --- | --- |
| $\wp(A) = \wp(B)$ | $A = B$ |
| $\wp(A) \subseteq \wp(B)$ | |
| $\forall Z \in \wp(A).\ Z \in \wp(B)$ | |
| $\wp(B) \subseteq \wp(A)$ | |
| $\forall Z \in \wp(B).\ Z \in \wp(A)$ | |

***Theorem:*** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

| *What I'm Assuming* | *What I Need to Show* |
|---|---|
| $\wp(A) = \wp(B)$ | ~~$A = B$~~ |
| $\wp(A) \subseteq \wp(B)$ | $A \subseteq B$ |
| $\forall Z \in \wp(A).\ Z \in \wp(B)$ | |
| $\wp(B) \subseteq \wp(A)$ | $B \subseteq A$ |
| $\forall Z \in \wp(B).\ Z \in \wp(A)$ | |

**Theorem:** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

| *What I'm Assuming* | *What I Need to Show* |
| --- | --- |
| $\wp(A) = \wp(B)$ | ~~$A = B$~~ |
| $\wp(A) \subseteq \wp(B)$ | ~~$A \subseteq B$~~ |
| $\forall Z \in \wp(A).\ Z \in \wp(B)$ | $\forall x \in A.\ x \in B.$ |
| $\wp(B) \subseteq \wp(A)$ | ~~$B \subseteq A$~~ |
| $\forall Z \in \wp(B).\ Z \in \wp(A)$ | $\forall z \in B.\ z \in A.$ |

**Theorem:** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

| *What I'm Assuming* | *What I Need to Show* |
|---|---|
| $\wp(A) = \wp(B)$ | ~~$A = B$~~ |
| $\wp(A) \subseteq \wp(B)$ | ~~$A \subseteq B$~~ |
| $\forall Z \in \wp(A).\ Z \in \wp(B)$ | ~~$\forall x \in A.$~~ $x \in B.$ |
| $\wp(B) \subseteq \wp(A)$ | ~~$B \subseteq A$~~ |
| $\forall Z \in \wp(B).\ Z \in \wp(A)$ | $\forall z \in B.\ z \in A.$ |
| $x \in A$ | |
| $\{x\} \subseteq A$ | |
| $\{x\} \in \wp(A)$ | |

***Theorem:*** Let $A$ and $B$ be sets. Then if $\wp(A) = \wp(B)$, then $A = B$.

| *What I'm Assuming* | | *What I Need to Show* |
|---|---|---|
| $\wp(A) = \wp(B)$ | | ~~$A = B$~~ |
| $\wp(A) \subseteq \wp(B)$ | | ~~$A \subseteq B$~~ |
| $\forall Z \in \wp(A).\ Z \in \wp(B)$ | | ~~$\forall x \in A.$~~ $x \in B.$ |
| $\wp(B) \subseteq \wp(A)$ | | ~~$B \subseteq A$~~ |
| $\forall Z \in \wp(B).\ Z \in \wp(A)$ | | $\forall z \in B.\ z \in A.$ |
| $x \in A$ | $x \in B$ | |
| $\{x\} \subseteq A$ | $\{x\} \subseteq B$ | |
| $\{x\} \in \wp(A)$ | $\{x\} \in \wp(B)$ | |

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:***

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$. This means that $\{x\} \in \wp(A)$, and since $\wp(A) \subseteq \wp(B)$ we know $\{x\} \in \wp(B)$.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$. This means that $\{x\} \in \wp(A)$, and since $\wp(A) \subseteq \wp(B)$ we know $\{x\} \in \wp(B)$. Thus we see that $\{x\} \subseteq B$, which in turn means that $x \in B$, as required.

***Theorem:*** Let $A$ and $B$ be sets. If $\wp(A) = \wp(B)$, then $A = B$.

***Proof:*** Assume $\wp(A) = \wp(B)$. We need to show that $A = B$, or, equivalently, that $A \subseteq B$ and $B \subseteq A$. Since the roles of $A$ and $B$ are symmetric, we'll just prove $A \subseteq B$.

Pick some $x \in A$; we need to show that $x \in B$. Since $x \in A$, we know that $\{x\} \subseteq A$. This means that $\{x\} \in \wp(A)$, and since $\wp(A) \subseteq \wp(B)$ we know $\{x\} \in \wp(B)$. Thus we see that $\{x\} \subseteq B$, which in turn means that $x \in B$, as required. ■